

Multi-tiling sets, Riesz bases, and sampling near the critical density in LCA groups

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Abstract

We prove the existence of sampling sets and interpolation sets near the critical density, in Paley Wiener spaces of a locally compact abelian (LCA) group G . This solves a problem left by Gröchenig, Kutyniok, and Seip in the article: ‘Landau’s density conditions for LCA groups’ (J. of Funct. Anal. 255 (2008) 1831-1850). To achieve this result, we prove the existence of universal Riesz bases of characters for $L^2(\Omega)$, provided that the relatively compact subset Ω of the dual group \widehat{G} satisfies a multi-tiling condition. This last result generalizes the Fuglede’s theorem, and extends to LCA groups setting recent constructions of Riesz bases of exponentials in bounded sets of \mathbb{R}^d .

Keywords: Sampling; Interpolation; Beurling’s densities; Riesz bases; Multi-tiling; Dyadic cubes; Locally compact abelian groups.

1 Introduction

Consider a locally compact abelian (LCA) group G , and let \widehat{G} denote its dual group. Given a relatively compact Borel subset Ω of \widehat{G} , the **Paley Wiener space** PW_Ω consists of all square integrable functions with Fourier transform supported on Ω . For this space, a set $\Lambda \subseteq G$ is a **sampling set** if there exist constants $A, B > 0$ such that for any $f \in PW_\Omega$,

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq B\|f\|_2^2.$$

On the other hand, Λ is an **interpolation set** for PW_Ω if the interpolation problem

$$f(\lambda) = c_\lambda,$$

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has a solution $f \in PW_\Omega$ for every $\{c_\lambda\}_{\lambda \in \Lambda} \in \ell^2(\Lambda)$. A set Λ that is at the same time a sampling and interpolation set is a **complete interpolation set**.

Using the Fourier transform, it turns out that Λ is a sampling set (resp. interpolation set, complete interpolation set) if and only if Λ , as a set set of characters restricted to Ω , is a frame (resp. Riesz sequence, Riesz basis) of $L^2(\Omega)$.

Sampling and interpolation sets satisfy the following necessary geometric conditions, proved by Landau in [13] for \mathbb{R}^d , and later on extended to LCA groups in [6]:

- A sampling set Λ for PW_Ω satisfies $\mathcal{D}^-(\Lambda) \geq m_{\widehat{G}}(\Omega)$;
- An interpolation set Λ for PW_Ω satisfies $\mathcal{D}^+(\Lambda) \leq m_{\widehat{G}}(\Omega)$,

where $m_{\widehat{G}}$ denotes the Haar measure of \widehat{G} , and \mathcal{D}^+ and \mathcal{D}^- denote the so called upper and lower Beurling densities (see Section 3.2 for precise definitions). In some sense, the Beurling's densities measure how the set Λ is distributed in G with respect to the distribution of a reference set, that for instance in the case of \mathbb{R}^d is the lattice \mathbb{Z}^d .

In [6], Gröchenig, Kutyniok and Seip raised the natural question of whether there exist sampling sets and interpolation sets for PW_Ω with densities arbitrarily close to the critical density $m_{\widehat{G}}(\Omega)$. Except for the particular case $G = \mathbb{R}^d$, proved by Marzo in [16] adapting a construction of Lyubarskii and Seip [15] and Kohlenberg [11], the problem remained open till now. The main obstacle, which is a recurrent problem in general LCA groups, is the absence of a natural substitute of rescalings. Therefore, a different approach is required.

The main goal of this paper is to give a complete solution to the aforementioned problem. A natural strategy is to show that, given a compact set Ω , there exists an outer (resp. inner) approximation set Ω_ε of Ω such that $L^2(\Omega_\varepsilon)$ has a Riesz basis of characters. As a consequence, we obtain the existence of sampling (resp. interpolation) sets near the critical density (see Theorem 3.5). Indeed, the Riesz basis of $L^2(\Omega_\varepsilon)$ becomes a frame for $L^2(\Omega)$ if Ω_ε is an outer approximation, and it becomes a Riesz sequence if Ω_ε is an inner approximation. The problem to accomplish this strategy in general LCA groups is to prove the existence of such “good” approximation sets. To overcome this difficulty, we proceed as follows.

First, we show that, given a relatively compact Borel set $\Omega \subset \widehat{G}$ that satisfies some tiling condition, the space $L^2(\Omega)$ admits a Riesz basis of characters (see Section 4 for details). This is motivated by recent results due to Grepstad and Lev in [5] (see also [12]) and it provides an extension of them. In order to prove this generalization we use operator theoretical techniques developed around the theory of shift invariant spaces. The shift invariant techniques provide a better understanding of the problem. As a consequence, besides the extension of the multi-tiling result of Grepstad and Lev to the group setting, we also prove a converse that is new even for \mathbb{R}^d (see Theorem 4.4). Furthermore, we answer negatively a question raised by Kolountzakis in [12].

In this way we get several candidates for approximation sets, i.e., those sets that satisfy a tiling condition with respect to a lattice. However, in general there are not sufficiently many of such sets to assure the required approximation, essentially because the group \widehat{G} may not have a rich family of lattices. In order to enlarge the family of candidates, we show that we can also consider sets Ω_ε that satisfy the tiling condition in an appropriate quotient group \widehat{G}/K instead of in the group \widehat{G} . Thus, we obtain a Riesz basis of characters in $L^2(\pi(\Omega_\varepsilon))$, where π denotes the canonical projection onto that quotient. Finally, we prove that this Riesz basis can be lifted to a Riesz basis of *characters* for $L^2(\Omega_\varepsilon)$ (see the last part of Section 5 for the details).

The paper is organized as follows. In section 2, we introduce preliminary results on LCA groups. Section 3 describes the main results. In section 4 we construct the Riesz basis of characters for $L^2(\Omega)$, under a multi-tiling condition on the set Ω . In section 5 we introduce the notion of quasi-dyadic cubes, which are used to construct the approximation sets. Finally, with all the necessary techniques at hand, we proceed to the proof of the main result in section 6.

2 Preliminaries

Throughout this section we review basic facts on locally compact abelian groups, (for more details see [2], [7], [8], [17]), setting in this way the notations we need for the following sections. Then, we introduce H -invariant spaces that generalize the concept of shift invariant spaces in the context of these groups (see [1]).

2.1 LCA Groups

Let G denote a Hausdorff locally compact abelian (LCA) group, and \widehat{G} its dual group, that is;

$$\widehat{G} = \{\gamma : G \rightarrow \mathbb{C}, \text{ and } \gamma \text{ is a continuous character of } G\},$$

where a character is a function satisfying the following properties:

- (i) $|\gamma(x)| = 1, \forall x \in G$;
- (ii) $\gamma(x + y) = \gamma(x)\gamma(y), \forall x, y \in G$.

Thus, the characters generalize the exponential functions $\gamma(x) = \gamma_t(x) = e^{2\pi itx}$ in the case $G = (\mathbb{R}, +)$. On every LCA group G there exists a Haar measure. It is a non-negative, regular Borel measure m_G that is non-identically zero and translation-invariant, which means:

$$m_G(E + x) = m_G(E),$$

for every element $x \in G$ and every Borel set $E \subset G$. This measure is unique up to a constant. Analogously to the Lebesgue spaces, we can define the $L^p(G) = L^p(G, m_G)$

spaces associated to the group G and the measure m_G :

$$L^p(G) := \left\{ f : G \rightarrow \mathbb{C}, f \text{ is measurable and } \int_G |f(x)|^p dm_G(x) < \infty \right\}.$$

Theorem 2.1. Let G be an LCA group and \widehat{G} its dual. Then

- (i) The dual group \widehat{G} , with the operation $(\gamma + \gamma')(x) = \gamma(x)\gamma'(x)$ is an LCA group. The topology in \widehat{G} is the one induced by the identification of the characters of the group with the characters of the algebra $L^1(G)$.
- (ii) The dual group of \widehat{G} is topologically isomorphic to G , that is, $\widehat{\widehat{G}} \approx G$, with the identification $g \in G \leftrightarrow e_g \in \widehat{\widehat{G}}$, where $e_g(\gamma) := \gamma(g)$.
- (iii) G is discrete (resp. compact) if and only if \widehat{G} is compact (resp. discrete).

As a consequence of (ii) of the previous theorem, we could use the notation (x, γ) for the complex number $\gamma(x)$, representing either the character γ applied to x or the character x applied to γ .

Taking $f \in L^1(G)$ we define the Fourier transform of f , as the function $\hat{f} : \widehat{G} \rightarrow \mathbb{C}$ given by

$$\hat{f}(\gamma) = \int_G f(x)(x, -\gamma) dm_G(x), \quad \gamma \in \widehat{G},$$

If the Haar measure of the dual group \widehat{G} is normalized conveniently, we obtain the inversion formula

$$f(x) = \int_{\widehat{G}} \hat{f}(\gamma)(x, \gamma) dm_{\widehat{G}}(\gamma),$$

for a specific class of functions. In the case that the Haar measures m_G and $m_{\widehat{G}}$ are normalized such that the inversion formula holds, the Fourier transform on $L^1(G) \cap L^2(G)$ can be extended to a unitary operator from $L^2(G)$ onto $L^2(\widehat{G})$. Thus the Parseval formula holds:

$$\langle f, g \rangle = \int_G f(x)\overline{g(x)} dm_G(x) = \int_{\widehat{G}} \hat{f}(\gamma)\overline{\hat{g}(\gamma)} dm_{\widehat{G}}(\gamma) = \langle \hat{f}, \hat{g} \rangle$$

for $f, g \in L^2(G)$. We conclude this subsection with the next classical result.

Proposition 2.2. If G is a compact group, then the characters of G form an orthonormal basis for $L^2(G)$.

2.2 H-invariant spaces

In this subsection we will review some basic aspects of the theory of shift invariant spaces in LCA groups. We will specially focus on the Paley Wiener spaces, that constitutes an important family of shift invariant spaces in which we are particularly interested. The

reader is referred to [1], where he can find the results in full generality, as well as other results related to shift invariant spaces in LCA groups. Let G be an LCA group, and let H be a **uniform lattice** on G , i.e., a discrete subgroup of G such that G/H is compact. Recall that a **Borel section** of G/H is a set of representatives of this quotient, that is, a subset A of G containing exactly one element of each coset. Thus, each element $x \in G$ has a unique expression of the form $x = a + h$ with $a \in A$ and $h \in H$. Moreover, it can be proved that there exists a relatively compact Borel section of G/H , which will be called **fundamental domain** (see [4] and [9]).

Definition 2.3. *We say that a closed subspace $V \subset L^2(G)$ is H -invariant if*

$$f \in V \text{ then } \tau_h f \in V, \forall h \in H,$$

where $\tau_h f(x) = f(x - h)$.

As we have mentioned, Paley Wiener spaces are important examples of H -invariant spaces, which in this context are defined by

$$PW_\Omega = \{f \in L^2(G) : \text{supp } \hat{f} \subset \Omega\},$$

where $\Omega \subset \widehat{G}$ is a relatively compact Borel set (see [6]). Actually, this space is invariant by any translation. In particular, it is H -invariant for any lattice H . Let Λ be the **dual lattice** of H ; that is, the **annihilator** of H defined by

$$\Lambda = \{\gamma \in \widehat{G} : (h, \gamma) = 1, \text{ for all } h \in H\}.$$

Suppose that Ω **tiles** \widehat{G} by translations of Λ , i.e.

$$\Delta_\Omega(x) := \sum_{\lambda \in \Lambda} \chi_\Omega(x - \lambda) = 1, \text{ a.e.}$$

In this case, it is well known that $\{e_h\}_{h \in H}$ is an orthonormal basis of $L^2(\Omega)$. Indeed, since Λ is also a uniform lattice, in particular, \widehat{G}/Λ is compact. So, as we recall in Proposition 2.2, $H \simeq (\widehat{G}/\Lambda)^\wedge$ is an orthonormal basis of $L^2(\widehat{G}/\Lambda)$. On the other hand, this space is isometrically isomorphic to $L^2(\Omega)$, because 1-tiling sets are Borel sections of the quotient group \widehat{G}/Λ up to a zero measure set.

In order to deal with multi-tiling sets, we recall that a set Ω **multi-tiles**, or more precisely **k -tiles** \widehat{G} by translations of Λ if

$$\Delta_\Omega(x) := \sum_{\lambda \in \Lambda} \chi_\Omega(x - \lambda) = k, \text{ a.e.}$$

For example, if Ω is a disjoint union of 1-tiling sets then the previous condition is satisfied. Next lemma shows that the reverse also holds, not only in \mathbb{R}^n (see Lemma 1 in [12]), but also in the context of the LCA groups.

Lemma 2.4. Let G be an LCA group and $H \subset G$ a countable discrete subgroup. A measurable set $\Omega \subset G$, k -tiles G under the translation set H , if and only if

$$\Omega = \Omega_1 \cup \dots \cup \Omega_k \cup R,$$

where R is a zero measure set, and the sets Ω_j , $1 \leq j \leq k$ are disjoint and each of them tiles G by translations of H .

Proof. If Ω is a disjoint union of k sets of representatives of G/H up to measure zero then clearly $\sum_{h \in H} \chi_\Omega(x - h) = k$, a.e.

For the converse, consider D to be a Borel section of G/H and let $\{h_j\}_{j \in \mathbb{N}}$ be an enumeration of the elements of H . We have $\Delta_\Omega(d) = k$ for almost all $d \in D$. If E denotes the set of the exceptions, define for $d \in D \setminus E$,

$$i_j(d) = \min\{n \in \mathbb{N} : \sum_{s=1}^n \chi_\Omega(d + h_s) = j\}, \quad j = 1, \dots, k.$$

and the measurable sets,

$$E_{j,n} = \{d \in D \setminus E : i_j(d) = n\}, \quad n \in \mathbb{N}.$$

Finally for $j = 1, \dots, k$, let $\Omega_j = \bigcup_{n \in \mathbb{N}} (E_{j,n} + h_n)$. It is straightforward to see that $\Omega = \bigcup_{j=1}^k \Omega_j \cup R$, is the desired decomposition. Here the remaining set $R = \Omega \setminus (\Omega_1 \cup \dots \cup \Omega_k)$ has measure zero because it is contained in $E + H$. \blacksquare

Let us recall now the following simple but useful proposition, that in the case of LCA groups is a direct consequence of Parseval identity and Weil's formula.

Proposition 2.5. Let D be a Borel section of \widehat{G}/Λ . The map $\mathcal{T} : L^2(G) \rightarrow L^2(D, \ell^2(\Lambda))$ defined by

$$\mathcal{T}f(\omega) = \{\hat{f}(\omega + \lambda)\}_{\lambda \in \Lambda},$$

is an isometric isomorphism. Moreover, for each element $h \in H$

$$\mathcal{T}(\tau_h f)(\omega) = e_h(\omega) \{\hat{f}(\omega + \lambda)\}_{\lambda \in \Lambda},$$

for almost every $\omega \in D$.

Remark 2.6. It is not difficult to see that if \hat{f} and \hat{g} are equal almost everywhere, then for almost every $\omega \in D$

$$\{\hat{f}(\omega + \lambda)\}_{\lambda \in \Lambda} = \{\hat{g}(\omega + \lambda)\}_{\lambda \in \Lambda}.$$

This guaranties that \mathcal{T} is well defined, and justifies the evaluation of elements of $L^2(G)$. With respect to the second part of Proposition 2.5, roughly speaking, it says that \mathcal{T} diagonalizes the H -translations. \blacktriangle

When the H -invariant space is finitely generated, Proposition 2.5 allows to translate a problem in (infinite dimensional) H -invariant spaces, to simpler linear algebra problems in finite dimensional Hilbert spaces. Suppose for instance that Ω is a measurable subset of \widehat{G} that k -tiles \widehat{G} by translations of Λ . Then, for almost every $\omega \in D$ there exist precisely k elements $\lambda_1, \dots, \lambda_k$, of Λ such that $\omega + \lambda_j \in \Omega$, where $\lambda_j = \lambda_j(\omega)$. Otherwise we would contradict that Ω is a k -tiling set. This implies that the subspace

$$\begin{aligned} J_\Omega(\omega) &:= \{ \{ \widehat{f}(\omega + \lambda) \}_{\lambda \in \Lambda} : f \in PW_\Omega \} \\ &\simeq \{ \{ \widehat{f}(\omega + \lambda_j) \}_{j=1, \dots, k} : f \in PW_\Omega \} \end{aligned}$$

has dimension at most k . This remark together with Proposition 2.5 lead to the following result.

Theorem 2.7. Let Ω be a relatively compact, k -tiling subset of \widehat{G} . Given $\phi_1, \dots, \phi_k \in PW_\Omega$ we define

$$T_\omega = \begin{pmatrix} \widehat{\phi}_1(\omega + \lambda_1) & \dots & \widehat{\phi}_k(\omega + \lambda_1) \\ \vdots & \ddots & \vdots \\ \widehat{\phi}_1(\omega + \lambda_k) & \dots & \widehat{\phi}_k(\omega + \lambda_k) \end{pmatrix}$$

where the $\lambda_j = \lambda_j(\omega)$ for $j = 1, \dots, k$ are the k values of $\lambda \in \Lambda$ such that $\omega + \lambda \in \Omega$. Then, the following statements are equivalent:

- (i) The set $\Phi_H = \{ \tau_h \phi_j : h \in H, j = 1, \dots, k \}$ is a Riesz basis for PW_Ω .
- (ii) There exist $A, B > 0$ such that for almost every $\omega \in D$,

$$A \|x\|^2 \leq \|T_\omega x\|^2 \leq B \|x\|^2,$$

for every $x \in \mathbb{C}^k$.

Moreover, in this case the constants of the Riesz basis are

$$A = \inf_{\omega \in D} \|T_\omega^{-1}\|^{-1} \quad \text{and} \quad B = \sup_{\omega \in D} \|T_\omega\|.$$

For a sake of completeness, we will give a proof of this result adapted to our setting. For the proof in more general H -invariant spaces see [1].

Proof. Let D be a fundamental domain of \widehat{G}/Λ . Consider a family $\{a_{j,h}\}$ with finitely many non-zero terms, where $j = 1, \dots, k$ and $h \in H$. Using the Fourier transform and a Λ -periodization argument we get

$$\left\| \sum_{j,h} a_{j,h} \tau_h \phi_j \right\|_{L^2(G)}^2 = \int_D \sum_{j,\ell=1}^k m_j(\omega) \left(\sum_{\lambda \in \Lambda} \widehat{\phi}_j(\omega + \lambda) \overline{\widehat{\phi}_\ell(\omega + \lambda)} \right) \overline{m_\ell(\omega)} dm_{\widehat{G}}(\omega).$$

where $m_j = \sum_{h \in H} a_{j,h} e_{-h}$. For each j , the vector $\{\hat{\phi}_j(\omega + \lambda)\}$ has at most k non-zero coordinates. More precisely, the only coordinates that can be different from zero are those corresponding to the elements $\lambda_j(\omega) \in \Lambda$ considered in the matrix T_ω . So, if $m = (m_1, \dots, m_k)$ then

$$\begin{aligned} \left\| \sum_{j=1}^k \sum_{h \in H} a_{j,h} \tau_h \phi_j \right\|_{L^2(G)}^2 &= \int_D \langle T_\omega^* T_\omega m(\omega), m(\omega) \rangle_{\mathbb{C}^k} dm_{\widehat{G}}(\omega) \\ &= \int_D \|T_\omega m(\omega)\|_{\mathbb{C}^k}^2 dm_{\widehat{G}}(\omega). \end{aligned} \quad (1)$$

On the other hand

$$\int_D \|m(\omega)\|_{\mathbb{C}^k}^2 dm_{\widehat{G}}(\omega) = \sum_{j=1}^k \int_D |m_j(\omega)|^2 dm_{\widehat{G}}(\omega) = \sum_{j=1}^k \sum_{h \in H} |a_{j,h}|^2. \quad (2)$$

Combining (1), (2) and standard arguments of measure theory we get that (i) \implies (ii).

For the other implication, note that from (1) and (2) we immediately get that the family Φ_H is a Riesz sequence for PW_Ω . So, it only remains to prove that condition (ii) also implies that Φ_H is complete. With this aim, let $f \in PW_\Omega$, and suppose that $\langle f, \tau_h \phi_j \rangle = 0$ for every $h \in H$ and $j = 1, \dots, m$. By a Λ -periodization argument we get

$$0 = \langle \hat{f}, e_{-h} \hat{\phi}_j \rangle = \int_D \left(\sum_{\lambda \in \Lambda} \hat{f}(\omega + \lambda) \overline{\hat{\phi}_j(\omega + \lambda)} \right) e_h(\omega) dm_{\widehat{G}}(\omega).$$

Since $\{e_h\}_{h \in H}$ is an orthonormal basis for $L^2(\widehat{G}/\Lambda)$, then

$$\sum_{\lambda \in \Lambda} \hat{f}(\omega + \lambda) \overline{\hat{\phi}_j(\omega + \lambda)} = 0, \quad a.e. \omega m_{\widehat{G}}.$$

This implies that $T_\omega^*(\{f(\omega + \lambda_j)\}_{j=1}^k) = 0$, where $\lambda_j = \lambda_j(\omega)$. Thus, $\mathcal{T}(f) = 0$ a.e $m_{\widehat{G}}$. ■

3 Sampling and interpolation near the critical density

In this section we study sampling sets, and interpolation sets on PW_Ω , when Ω is a relatively compact subset of \widehat{G} . In [6], Gröchenig, Kutyniok, and Seip introduced two notions of densities that suitably generalize the Beurling's densities defined in \mathbb{R}^d . Our main goal is to prove that there exist sampling sets, and interpolation sets whose densities are arbitrarily close to the critical one, answering a question raised by Gröchenig et. al. in [6].

3.1 Standing hypothesis

Since we will work with relatively compact sets Ω , throughout this paper we will assume that G is an LCA group such that its dual \widehat{G} is compactly generated (but not compact to avoid trivialities). By the standard structure theorems, \widehat{G} is isomorphic to $\mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{K}$, where \mathbb{K} is a compact subgroup of \mathbb{T}^ω . Consequently, G is isomorphic to $\mathbb{R}^d \times \mathbb{T}^m \times \mathbb{D}$, where \mathbb{D} is a countable discrete group. In particular, note that both G and \widehat{G} are second countable.

This is not a serious restriction, as the following lemma shows (see [3] or [6]).

Lemma 3.1. Assume that $\Omega \subseteq \widehat{G}$ is relatively compact, and let H be the open subgroup generated by Ω . Then H is compactly generated and there exists a compact subgroup $K \subseteq G$ such that every $f \in PW_\Omega$ is K -periodic. Furthermore, the quotient G/K is homeomorphic to $\mathbb{R}^d \times \mathbb{T}^m \times \mathbb{D}$, where \mathbb{D} is a countable discrete abelian group, and $\widehat{G/K} \simeq H$.

Therefore, given a relatively compact set $\Omega \subseteq \widehat{G}$, this lemma shows that the space PW_Ω essentially lives in $L^2(G/K)$, and $\widehat{G/K} \simeq H$ is compactly generated.

3.2 Beurling-type densities in LCA groups

To begin with, recall that a subset Λ of G is called **uniformly discrete** if there exists an open set U such that the sets $\lambda + U$ (λ in Λ) are pairwise disjoint. In some sense, the densities in \mathbb{R}^d compare the concentration of the points of a given discrete set with that of the integer lattice \mathbb{Z}^d . In a topological group, this comparison is done by means of the following relation:

Definition 3.2. Given two uniformly discrete sets Λ and Λ' and non-negative numbers α and α' , we write $\alpha\Lambda \preccurlyeq \alpha'\Lambda'$ if for every $\varepsilon > 0$ there exists a compact subset K of G such that for every compact subset L we have

$$(1 - \varepsilon)\alpha \#(\Lambda \cap L) \leq \alpha' \#(\Lambda' \cap (K + L)).$$

Now, we have to fix a reference lattice in the group G . As we mentioned at the beginning of this section, since \widehat{G} is compactly generated, G is isomorphic to $\mathbb{R}^d \times \mathbb{T}^m \times \mathbb{D}$, where \mathbb{D} is a countable discrete group. So, a natural reference lattice is $H_0 = \mathbb{Z}^d \times \{e\} \times \mathbb{D}$. Using this reference lattice, and the above transitive relation, we have all what we need to recall the definitions of upper and lower densities.

Definition 3.3. Let Λ be a uniformly discrete subset of G . The **lower uniform density** of Λ is defined as

$$\mathcal{D}^-(\Lambda) = \sup\{\alpha \in \mathbb{R}^+ : \alpha H_0 \preccurlyeq \Lambda\}.$$

On the other hand, its **upper uniform density** is

$$\mathcal{D}^+(\Lambda) = \inf\{\alpha \in \mathbb{R}^+ : \Lambda \preccurlyeq \alpha H_0\}.$$

These densities always satisfy that $\mathcal{D}^-(\Lambda) \leq \mathcal{D}^+(\Lambda)$, and they are finite. Moreover, it can be shown that the infimum and the supremum are actually a minimum and a maximum. In the case that both densities coincide, we will simply write $\mathcal{D}(\Lambda)$. It should be also mentioned that in the case of \mathbb{R}^d , these densities coincide with the Beurling's densities when the reference lattice is \mathbb{Z}^d .

Using these densities, Gröchenig, Kutyniok, and Seip obtained in [6] the following extension of the classical result of Landau to LCA groups.

Theorem 3.4. Suppose Λ is a uniform discrete subset of G . Then

- S) If Λ is a sampling set for PW_Ω , then $\mathcal{D}^-(\Lambda) \geq m_{\widehat{G}}(\Omega)$;
- I) If Λ is an interpolation set for PW_Ω , then $\mathcal{D}^+(\Lambda) \leq m_{\widehat{G}}(\Omega)$.

A natural question is whether there exist sampling sets and interpolation sets near the critical density. In the case $G = \mathbb{R}^d$ a positive answer was given by Marzo in [16]. The following theorem is our main result, which completely answers this question.

Theorem 3.5. Let Ω be a compact subset of \widehat{G} , and let $\varepsilon > 0$. Then, the following statements hold:

- (i) There exists a sampling set J_ε for PW_Ω such that

$$\mathcal{D}(J_\varepsilon) \leq m_{\widehat{G}}(\Omega) + \varepsilon.$$

- (ii) If $m_{\widehat{G}}(\partial\Omega) = 0$, then there exists an interpolation set J^ε for PW_Ω such that

$$\mathcal{D}(J^\varepsilon) \geq m_{\widehat{G}}(\Omega) - \varepsilon.$$

Although roughly speaking the strategy of the proof will be similar to the one used in [15] and [16] (see also [11]), in order to pursue this strategy, we will have to overcome several technical issues. This will be done in the following two sections. Finally, in section 6 we will combine the obtained results, and we will provide the proof of Theorem 3.5.

4 Constructing Riesz basis in the context of LCA groups

The relation between multi-tiling sets and the existence of Riesz bases in the \mathbb{R}^d setting was firstly pointed out in [5] by Grepstad-Lev. More precisely, they proved that a bounded Riemann integrable Borel set $\Omega \subseteq \mathbb{R}^d$ admits a Riesz basis of exponentials if it multi-tiles \mathbb{R}^d with translation set a lattice Λ . Later on, Kolountzakis gave in [12] a simpler proof of this result in a slightly more general form (see also [10] for a different approach). Important special cases had been proved by Lyubarskii-Seip in [15], and Marzo in [16], (see also [14] and [18]).

One of the main theorems of this section is the following generalization of Grepstad-Lev's result to the LCA group setting.

Theorem 4.1. Let H be a uniform lattice of G , Λ its dual lattice, and $k \in \mathbb{N}$. Then, there exist $a_1, \dots, a_k \in G$, depending only on the lattice Λ , such that for any relatively compact Borel subset Ω of \widehat{G} satisfying

$$\Delta_\Omega(\omega) := \sum_{\lambda \in \Lambda} \chi_\Omega(\omega - \lambda) = k, \quad \text{a.e. } \omega \in \widehat{G},$$

the set

$$\{e_{a_j - h} \chi_\Omega : h \in H, j = 1, \dots, k\}$$

is a Riesz basis for $L^2(\Omega)$.

We would like to emphasize that, in the previous theorem, the same set $\{a_1, \dots, a_k\}$ can be used for any k -tiling set Ω . If we call such a k -tuple (a_1, \dots, a_k) **H -universal**. The following result is a slight improvement of the already known results.

Theorem 4.2. Let H be a uniform lattice of G and $k \in \mathbb{N}$. Then, there exists a Borel set $N \subseteq G^k$ such that $m_{G^k}(N) = 0$ and every k -tuple $(a_1, \dots, a_k) \in G^k \setminus N$ is H -universal.

Remark 4.3. Note that, if we fix a fundamental domain D , given any universal k -tuple (a_1, \dots, a_k) there exists a unique k -tuple $(d_1, \dots, d_k) \in D^k$ such that

$$\{e_{a_j - h} \chi_\Omega : h \in H, j = 1, \dots, k\} = \{e_{d_j - h} \chi_\Omega : h \in H, j = 1, \dots, k\}.$$

So, we can restrict our attention to universal k -tuples belonging to D^k . In this case, consider the ‘‘uniform’’ probability measure on D^k given by the restriction of the Haar measure of G^k to D^k (conveniently normalized). Then another way to state Theorem 4.2 is that a k -tuple in D^k is almost surely H -universal. \blacktriangle

The second main result of this section is the following kind of converse of Theorem 4.1.

Theorem 4.4. Let H be a uniform lattice of G and Λ its dual lattice. Given a relatively compact subset Ω of \widehat{G} , if $L^2(\Omega)$ admits a Riesz basis of the form

$$\{e_{a_j - h} \chi_\Omega : h \in H, j = 1, \dots, k\}$$

then Ω k -tiles \widehat{G} .

The proofs of these results are provided in the next subsection. Then, in the last subsection we will show a counterexample that answers negatively a question raised by Kolountzakis in [12].

4.1 Proofs of Theorems 4.1, 4.2 and 4.4

We begin with two technical lemmas. Following Rudin's book [17], we will say that a function p is a **trigonometric polynomial** on G if it has the form

$$p(g) = \sum_{j=0}^n c_j \gamma_j(g)$$

for some $n \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $\gamma_j \in \widehat{G}$.

Lemma 4.5. The zero set of a trigonometric polynomial p on G has zero Haar measure.

Proof. By the standing hypothesis, we can identify G with the group $\mathbb{R}^d \times \mathbb{T}^m \times \mathbb{D}$, for some countable discrete LCA group \mathbb{D} . Hence, given $(x, \omega, d) \in \mathbb{R}^d \times \mathbb{T}^m \times \mathbb{D}$, the polynomial p can be written as

$$p(x, \omega, d) = \sum_{j=0}^n c_j \rho_j(x) \tau(\omega) \delta(d)$$

where $\rho \in \widehat{\mathbb{R}^d}$, $\tau \in \widehat{\mathbb{T}^m}$, and $\delta \in \widehat{\mathbb{D}}$. Let $C_p = \{(x, \omega, d) : p(x, \omega, d) = 0\}$, and suppose by contradiction that $m_G(C_p) > 0$. Since

$$C_p = \bigcup_{d \in D} C_p \cap (\mathbb{R}^d \times \mathbb{T}^m \times \{d\}),$$

there exists $d_0 \in D$ such that

$$m_G(C_p \cap (\mathbb{R}^d \times \mathbb{T}^m \times \{d_0\})) > 0$$

If we restrict p to $\mathbb{R}^d \times \mathbb{T}^m \times \{d_0\}$ we get the trigonometric polynomial q on $\mathbb{R}^d \times \mathbb{T}^m$

$$q(x, \omega) = \sum_{j=0}^n (c_j \delta(d_0)) \rho_j(x) \tau(\omega)$$

that is non-trivial and its zero set has positive measure. This is a contradiction, and therefore $m_g(C_p) = 0$. ■

Lemma 4.6. Let K_1 and K_2 be compact subsets of \widehat{G} . If

$$\Gamma = \{\lambda \in \Lambda : (\lambda + K_1) \cap K_2 \neq \emptyset\},$$

then $\#\Gamma < \infty$.

Proof. Note that $\Gamma \subset \Lambda \cap (K_1 - K_2)$, where $K_1 - K_2 = \{k_1 - k_2 : k_j \in K_j, j = 1, 2\}$. Since Λ is a discrete set and $(K_1 - K_2)$ is compact, Γ should be necessarily a finite set. ■

Proof of Theorems 4.1 and 4.2. Given $a_1, \dots, a_k \in G$, define the functions ϕ_1, \dots, ϕ_k by their Fourier transform in the following way:

$$\hat{\phi}_j := e_{a_j} \chi_\Omega, \quad j \in \{1, \dots, k\}. \quad (3)$$

We will show that under the hypothesis on Ω , there exist a_1, \dots, a_k such that ϕ_1, \dots, ϕ_k translated by H form a Riesz basis for PW_Ω .

Choose a fundamental domain D of \widehat{G}/Λ . Since Ω is a set that k -tiles \widehat{G} , for almost every $\omega \in D$, the vectors $\widehat{\phi}_j(\omega)$ have at most k entries different from zero. These entries are those that correspond to the (different) elements $\lambda_j = \lambda_j(\omega) \in \Lambda$, $1 \leq j \leq k$, such that $\omega + \lambda_j \in \Omega$. For $\omega \in D$ consider the matrix

$$T_\omega = \begin{pmatrix} \widehat{\phi}_1(\omega + \lambda_1) & \dots & \widehat{\phi}_k(\omega + \lambda_1) \\ \vdots & \ddots & \vdots \\ \widehat{\phi}_1(\omega + \lambda_k) & \dots & \widehat{\phi}_k(\omega + \lambda_k) \end{pmatrix}$$

By Theorem 2.7, the H -translations of ϕ_1, \dots, ϕ_k form a Riesz basis for PW_Ω if and only if there exist $A, B > 0$ such that

$$A\|x\|^2 \leq \|T_\omega x\|^2 \leq B\|x\|^2, \quad (4)$$

for every $x \in \mathbb{C}^k$ and almost every $\omega \in D$. The rest of the proof follows ideas of [12] suitably adapted to our setting. Firstly, note that

$$\begin{aligned} T_\omega &= \begin{pmatrix} \widehat{\phi}_1(\omega + \lambda_1) & \dots & \widehat{\phi}_k(\omega + \lambda_1) \\ \vdots & \ddots & \vdots \\ \widehat{\phi}_1(\omega + \lambda_k) & \dots & \widehat{\phi}_k(\omega + \lambda_k) \end{pmatrix} = \begin{pmatrix} e_{a_1}(\omega + \lambda_1) & \dots & e_{a_k}(\omega + \lambda_1) \\ \vdots & \ddots & \vdots \\ e_{a_1}(\omega + \lambda_k) & \dots & e_{a_k}(\omega + \lambda_k) \end{pmatrix} \\ &= \begin{pmatrix} (a_1, \lambda_1) & \dots & (a_k, \lambda_1) \\ \vdots & \ddots & \vdots \\ (a_1, \lambda_k) & \dots & (a_k, \lambda_k) \end{pmatrix} \begin{pmatrix} (a_1, \omega) & 0 & \dots & 0 & 0 \\ 0 & (a_2, \omega) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & (a_{k-1}, \omega) & 0 \\ 0 & 0 & \dots & 0 & (a_k, \omega) \end{pmatrix} = E_\omega U_\omega. \end{aligned} \quad (5)$$

Since U_ω is unitary, to prove the inequalities in (4) is equivalent to show that

$$A\|x\|^2 \leq \|E_\omega x\|^2 \leq B\|x\|^2, \quad (6)$$

for every $x \in \mathbb{C}^k$, and almost every $\omega \in D$. By Lemma 4.6, applied to $K_1 = \overline{D}$ and $K_2 = \overline{\Omega}$, when ω runs over (a full measure subset of) the Borel section D , only a finite number of different matrices E_ω appear in (5), say E_1, \dots, E_N . Thus, it is enough to prove that they are all invertible. Note that the determinants of the E_ω are polynomials of the form

$$d(x_1, \dots, x_k) = \sum_{\pi \in S_k} \operatorname{sgn}(\pi) \prod_{j=1}^k (x_{\pi(j)}, \lambda_j(\omega)),$$

evaluated in $(a_1, \dots, a_k) \in G \times \dots \times G = G^k$, where S_k denotes the permutation group on $1, \dots, k$. Since Λ is countable, the set of trigonometric polynomials on G^k

$$\mathcal{P}_k = \left\{ p(x_1, \dots, x_k) = \sum_{\pi \in S_k} \text{sgn}(\pi) \prod_{j=1}^k (x_{\pi(j)}, \lambda_j) : \text{for any } (\lambda_1, \dots, \lambda_k) \in \Lambda^k \right\}$$

is countable. This set contains the trigonometric polynomials $d(x_1, \dots, x_k)$ associated to the determinants of the matrices E_j . Note that it also contains the polynomials associated to matrices E'_j coming from any other k -tiling set. Therefore, the universal k -tuple (a_1, \dots, a_k) that we are looking for, is any k -tuple such that

$$p(a_1, \dots, a_k) \neq 0 \quad \forall p \in \mathcal{P}_k.$$

To prove that such a k -tuple exists, we will use a measure theoretical argument based on Lemma 4.5. Note that G^k is a compactly generated LCA group, and Λ^k is the dual lattice of the uniform lattice H^k in G^k . Hence, using Lemma 4.5 with G^k instead of G , we get that the union of the zero sets corresponding to these polynomials has zero Haar measure in G^k . Therefore, there exist $a_1, \dots, a_N \in G$ so that (a_1, \dots, a_N) does not belong to any of these zero sets. In particular, for these values of a_j , the matrices E_1, \dots, E_N are invertible. Then, by Theorem 2.7, the H -translations of the functions ϕ_1, \dots, ϕ_k form a Riesz basis for PW_Ω . This is equivalent to say that

$$\{e_{a_j-h} \chi_\Omega : h \in H, j = 1, \dots, k\},$$

is a Riesz basis on $L^2(\Omega)$. The same holds for any other k -tiling set Ω' by construction of \mathcal{P}_k and the k -tuple (a_1, \dots, a_k) . \blacksquare

Proof of Theorem 4.4. Let D be a fundamental domain of \widehat{G}/Λ . Since both D and Ω are relatively compact, by Lemma 4.6 the set $\Lambda_0 := \{\lambda \in \Lambda : (\lambda + D) \cap \Omega \neq \emptyset\}$ is finite. Recall that for each $\omega \in D$

$$J_\Omega(\omega) = \{ \{f(\omega + \lambda)\}_{\lambda \in \Lambda} : f \in L^2(\Omega) \}.$$

Then, the hypothesis implies that there exists a measurable set $E \subseteq D$ of zero Haar measure such that for every $\omega \in D \setminus E$ the set of vectors

$$\{e_{a_j}(\omega + \lambda) \chi_\Omega(\omega + \lambda)\}_{\lambda \in \Lambda} \tag{7}$$

is a Riesz basis of $J_\Omega(\omega)$. In particular, for every $\omega \in D \setminus E$ we have that

$$\#\{\lambda \in \Lambda_0 : \omega + \lambda \in \Omega\} \geq k.$$

Indeed, otherwise the dimension of $J_\Omega(\omega)$ would be smaller than k , which contradicts the fact that the k vectors in (7) form a Riesz basis of $J_\Omega(\omega)$. Now, assume that there exists a set of positive measure $F \subseteq D \setminus E$ such that for every $\omega \in F$

$$\#\{\lambda \in \Lambda_0 : \omega + \lambda \in \Omega\} > k.$$

Since Λ_0 is finite, the set $\{\Gamma \subset \Lambda_0 : \#\Gamma > k\}$ is also finite. So, there exists $F_0 \subseteq F$ of positive measure and $\lambda_1, \dots, \lambda_n \in \Lambda_0$, $n > k$, such that for every $\omega \in F_0$ it holds that $\omega + \lambda_j \in \Omega$. Note that this implies that for every λ_j

$$\lambda_j + F_0 \subseteq \Omega.$$

On the other hand, as D is a fundamental domain, the sets $\lambda_1 + F_0, \dots, \lambda_n + F_0$ are disjoint. Now consider the characteristic functions $f_j = \chi_{\lambda_j + F_0}$ for each $j \in \{1, \dots, n\}$. All these functions belongs to $L^2(\Omega)$, and therefore for every $j \in \{1, \dots, k\}$

$$\{f_j(\omega + \lambda)\}_{\lambda \in \Lambda} \in J_\Omega(\omega).$$

Note that $f_j(\omega + \lambda) = \delta_{\lambda, \lambda_j}$. This implies that for every $\omega \in F_0$ the dimension of $J_\Omega(\omega)$ is $n > k$, which is not possible because the set of vector in (7) is a Riesz basis for $J_\Omega(\omega)$. In conclusion, for almost every $w \in D$

$$\#\{\lambda \in \Lambda_0 : \omega + \lambda \in \Omega\} = k,$$

which precisely says that Ω is k -tiling. ■

4.2 A counterexample for unbounded sets in \mathbb{R}

The same scheme can not be applied if Ω is not relatively compact, as the following example shows. This example also gives a negative answer to the open problem left by Kolountzakis in [12].

Example 4.7. Let $G = \mathbb{R}$, and consider the following subset of $\widehat{\mathbb{R}} \simeq \mathbb{R}$:

$$\Omega_0 = [0, 1) \cup \bigcup_{n=2}^{\infty} [n - 2^{-(n-2)}, n - 2^{-(n-1)}).$$

This is a 2-tiling set with respect to the lattice \mathbb{Z} (see Figure 1).

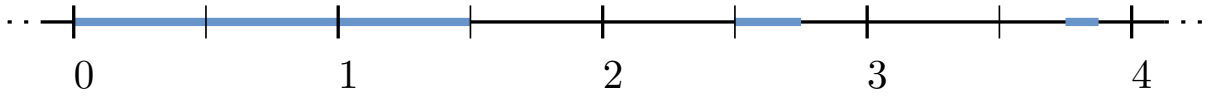


Figure 1: The set Ω_0 .

However, if we consider the functions ϕ_1 and ϕ_2 defined through their Fourier transform by

$$\hat{\phi}_j(\omega) = e^{2\pi i a_j \omega} \chi_\Omega,$$

for $j = 1, 2$, then integer translations of ϕ_1 and ϕ_2 are not a Riesz basis for PW_Ω for any choice of $a_1, a_2 \in \mathbb{R}$. In other words, $(a_1 + \mathbb{Z}) \cup (a_2 + \mathbb{Z})$ is not a complete interpolation sets for PW_Ω for any pair $a_1, a_2 \in \mathbb{R}$.

To show this, recall that in the proof of Theorem 4.1 we proved that the integer translations of ϕ_1 and ϕ_2 form a Riesz basis for PW_Ω if and only if the matrices

$$E_\omega = \begin{pmatrix} e^{2\pi i a_1 \lambda_1(\omega)} & e^{2\pi i a_2 \lambda_1(\omega)} \\ e^{2\pi i a_1 \lambda_2(\omega)} & e^{2\pi i a_2 \lambda_2(\omega)} \end{pmatrix},$$

and their inverses are uniformly bounded for almost every ω in the fundamental domain, that for simplicity we choose the interval $[0, 1)$. For this particular Ω_0 , $\lambda_1(\omega)$ is always equal to zero, while $\lambda_2(\omega) = n$ if $\omega \in [1 - 2^{-(n-1)}, 1 - 2^{-n})$, for $n \in \mathbb{N}$. Therefore

$$E_\omega = \begin{pmatrix} 1 & 1 \\ e^{2\pi i a_1 \lambda_2(\omega)} & e^{2\pi i a_2 \lambda_2(\omega)} \end{pmatrix},$$

which can be rewritten as

$$E_\omega = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i a_1 \lambda_2(\omega)} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & e^{2\pi i (a_2 - a_1) \lambda_2(\omega)} \end{pmatrix}.$$

So, if $a_2 - a_1 \in \mathbb{Q}$, there exists a set of positive measure such that the matrices T_ω are not invertible for ω in that set. On the other hand, if $a_2 - a_1 \notin \mathbb{Q}$, as the set $\{e^{2\pi i (a_2 - a_1)n}\}_{n \in \mathbb{N}}$ is dense in \mathbb{T} , the matrices E_ω^{-1} are not uniformly bounded.

Since the set Ω_0 also multi-tiles \mathbb{R} with other lattices too, a natural question is whether or not we can obtain Riesz basis using these lattices. The answer is no, and the idea of the proof is essentially the same. For this reason, we only make some comments on the main differences, and we leave the details to the reader. First of all, recall that the (uniform) lattices of \mathbb{R} have the form $\Lambda_\alpha = \alpha\mathbb{Z}$, for some $\alpha \in \mathbb{R}$. It is not difficult to prove that Ω_0 multi-tiles \mathbb{R} only for lattices corresponding to $\alpha = k^{-1}$, for some $k \in \mathbb{N}$. Moreover, with respect to the lattice $\Lambda_{k^{-1}}$, the set Ω_0 is $2k$ -tiling. Given $a_1, \dots, a_{2k} \in \mathbb{R}$, as we already mentioned, in the proof of Theorem 4.1 we show that $(a_1 + \mathbb{Z}) \cup \dots \cup (a_{2k} + \mathbb{Z})$ is a complete interpolation sets for PW_Ω if and only if the matrices

$$E_\omega = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ e^{2\pi i a_1 \lambda_1(\omega)} & e^{2\pi i a_2 \lambda_1(\omega)} & \dots & e^{2\pi i a_{2k-1} \lambda_1(\omega)} & e^{2\pi i a_{2k} \lambda_1(\omega)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e^{2\pi i a_1 \lambda_{2k-1}(\omega)} & e^{2\pi i a_2 \lambda_{2k-1}(\omega)} & \dots & e^{2\pi i a_{2k-1} \lambda_{2k-1}(\omega)} & e^{2\pi i a_{2k} \lambda_{2k-1}(\omega)} \\ e^{2\pi i a_1 \lambda_{2k}(\omega)} & e^{2\pi i a_2 \lambda_{2k}(\omega)} & \dots & e^{2\pi i a_{2k-1} \lambda_{2k}(\omega)} & e^{2\pi i a_{2k} \lambda_{2k}(\omega)} \end{pmatrix}$$

and their inverses are uniformly bounded for almost every $\omega \in [0, k^{-1})$. By construction of Ω_0 , for each $m \in \mathbb{N}$ we can find an interval $I \subseteq [0, k^{-1})$ of positive measure such that for every $\omega \in I$ there exists $j \in \{1, \dots, 2k\}$ so that $\lambda_j(\omega) = m$. On the other hand, as in the case studied before, either the orbit

$$\{(e^{2\pi i a_1 m}, e^{2\pi i a_2 m}, \dots, e^{2\pi i a_{2k-1} m}, e^{2\pi i a_{2k} m})\}_{m \in \mathbb{Z}},$$

is periodic or there exists elements of the orbit as close as we want to the first row vector $(1, \dots, 1)$. Therefore, the matrices E_ω^{-1} can not be uniformly bounded in a full measure subset of $[0, 1)$.

5 Quasi-dyadic cubes

In the previous section we proved that there exist many Borel sets Ω in \widehat{G} such that $L^2(\Omega)$ admits a Riesz basis of characters. However, in general, they are not enough to approximate any compact set. Let us briefly explain the reason. In the classical case of \mathbb{R}^d , the approximation is done by means of sets that are union of dyadic cubes. Note that the dyadic cubes of side length equal to 2^{-n} are fundamental domains for the lattice $2^{-n}\mathbb{Z}$. Hence, the dilation of the cubes is reflected in the refinement of the lattices. So, in order to get a good approximation in more general LCA groups, the idea is to look for a nested family of lattices whose corresponding fundamental domains become in some sense smaller and smaller. This is the main issue, because to get such family in the group $\widehat{G} \simeq \mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{T}^\ell$ may be difficult (or even impossible), because of the compact factor. The key to overcome this difficulty is given by the following classical result (see [7]).

Lemma 5.1. Given a neighborhood U of e in \widehat{G} , there exists a compact subgroup K included in U and such that \widehat{G}/K is elemental, that is

$$\widehat{G}/K \simeq \mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{T}^\ell \times F, \quad (8)$$

where F is a finite group.

In $\mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{T}^\ell \times F$, the above mentioned strategy to obtain dilations by means of refinements of lattices can be done without any problem. More precisely, we consider

$$\begin{aligned} \Lambda_n &= \Lambda_n(d, m, \ell) = (2^{-n}\mathbb{Z})^d \times \mathbb{Z}^m \times \mathbb{Z}_{2^n}^\ell \times F \subseteq \widehat{G}/K, \\ \mathcal{Q}_0^{(n)} &= [-2^{-n-1}, 2^{-n-1}]^d \times \{0\} \times [-2^{-n-1}, 2^{-n-1}]^\ell \times \{e\}. \end{aligned} \quad (9)$$

This leads to the following definition of quasi-dyadic cubes.

Definition 5.2. Let K be a compact subgroup of \widehat{G} such that \widehat{G}/K is elemental, and let π the canonical projection from \widehat{G} onto the quotient. Identifying the quotient \widehat{G}/K with the group $\mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{T}^\ell \times F$, the family of **quasi-dyadic cubes of generation n associated to K** , denoted by $\mathcal{D}_K^{(n)}$, are defined by

$$Q_\lambda^{(n)} = \pi^{-1}(\mathcal{Q}_\lambda^{(n)})$$

where $\mathcal{Q}_\lambda^{(n)} = \lambda + \mathcal{Q}_0^{(n)}$ for $\lambda \in \Lambda_n$.

Note that in order to distinguish the cubes in the quotient from the cubes in \widehat{G} , for those in \widehat{G}/K we use calligraphic letters. Note also that the quasi-dyadic cubes $Q_\lambda^{(n)}$ are relatively compact. Indeed, if $S_\lambda^{(n)}$ is a relatively compact Borel section of $\mathcal{Q}_\lambda^{(n)}$ in the group \widehat{G} , then $Q_\lambda^{(n)} = S_\lambda^{(n)} + K$.

The main difference with the classical case, is that the quasi-dyadic cubes are parametrized not only by a (dyadic) lattice, but also by some compact subgroups. This family of quasi-dyadic cubes clearly satisfies many of the arithmetical and combinatorial properties of the classical dyadic cubes. However, for our purposes, the following approximation result is the most important.

Proposition 5.3. Let C be a compact set and V an open set such that $C \subset V \subset \widehat{G}$. There exists a compact subgroup K of \widehat{G} such that \widehat{G}/K is an elemental LCA group, and $Q_{\lambda_1}^{(m)}, \dots, Q_{\lambda_k}^{(m)} \in \mathcal{D}_K^{(m)}$ for $m \in \mathbb{N}$ large enough such that

$$C \subseteq \bigcup_{j=1}^k Q_{\lambda_j}^{(m)} = \pi^{-1} \left(\bigcup_{j=1}^k \lambda_j + \mathcal{Q}_0^{(m)} \right) \subseteq V.$$

Proof. Let U be a compact neighborhood of e in \widehat{G} such that $C + U \subseteq V$. Take a compact subgroup K contained in U which satisfies that

$$\widehat{G}/K \simeq \mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{T}^\ell \times F,$$

for some integers $d, m, \ell \geq 0$. Let $\pi : \widehat{G} \rightarrow \widehat{G}/K$ denote the canonical projection. By our assumptions on the open set U we have for n large enough that

$$C \subseteq C + Q_0^{(2n)} \subseteq C + Q_0^{(n)} \subseteq V.$$

On the other hand, by the compactness of C , there exist $\gamma_1, \dots, \gamma_j \in C$ such that

$$C \subseteq \bigcup_{i=1}^j (\gamma_i + Q_0^{(2n)}).$$

Consider the lattice Λ_{4n} , and $\lambda_{i,1}, \dots, \lambda_{i,s_i} \in \Lambda_{4n}$ such that

$$\pi(\gamma_i + Q_0^{(2n)}) = \pi(\gamma_i) + \mathcal{Q}_0^{(2n)} \subseteq \bigcup_{h=1}^{s_i} (\lambda_{i,h} + \mathcal{Q}_0^{(4n)}) \subseteq \pi(\gamma_i) + \mathcal{Q}_0^{(n)}.$$

Let $\{\lambda_1, \dots, \lambda_k\}$ an enumeration of the elements of Λ_{4n} used to cover all the sets $\pi(\gamma_i) + \mathcal{Q}_0^{(2n)}$. Then, the above inclusions imply that

$$C \subseteq \bigcup_{i=1}^j (\gamma_i + Q_0^{(2n)}) \subseteq \bigcup_{j=1}^k \pi^{-1}(\lambda_j + \mathcal{Q}_0^{(4n)}) \subseteq \bigcup_{i=1}^j (\gamma_i + Q_0^{(n)}) \subset V$$

Thus, we can take $m = 4n$, and the proof is complete. ■

Another good property of the quasi-dyadic cubes is the following.

Proposition 5.4. Let Ω be finite a union of quasi-dyadic cubes in $\mathcal{D}_K^{(n)}$. Then, the space $L^2(\Omega)$ admits a Riesz basis of characters (restricted to Ω).

If $\pi : \widehat{G} \rightarrow \widehat{G}/K$ denotes the canonical projection, then the set $\pi(\Omega)$ multi-tiles the quotient space with the lattice Λ_n defined in (9). By Theorem 4.1, this implies the existence of a Riesz basis of characters in the space $L^2(\pi(\Omega))$. Now, we need a result that gives us a way to lift this basis. This is provided by the following result, which in particular concludes the proof of Proposition 5.4.

Theorem 5.5. Let K be a compact subgroup of an LCA group G such that \widehat{K} is countable. Suppose that there exists a subset Q of G/K such that $L^2(Q)$ admits a Riesz basis of characters of G/K . If $\pi : G \rightarrow G/K$ denotes the canonical projection, and $\widetilde{Q} = \pi^{-1}(Q)$, then $L^2(\widetilde{Q})$ also admits a Riesz basis of characters.

Proof. On the one hand, note that $\widehat{G/K} \simeq K^\perp \subseteq \widehat{G}$, where K^\perp denotes the annihilator of K . So, the Riesz basis for $L^2(Q)$ can be identified with some elements $\{\gamma_n\}$ in \widehat{G} . On the other hand, the elements \widehat{K} form an orthonormal basis for $L^2(K)$ endowed with the normalized Haar measure m_K . Moreover, since \widehat{K} can be identified with the quotient group \widehat{G}/K^\perp , the orthonormal basis for $L^2(K)$ can be identified with a system of representatives $\{\kappa_m\}$ of \widehat{G}/K^\perp .

Now, we will prove that $\{\gamma_n + \kappa_m\}$ is a Riesz basis for $L^2(\widetilde{Q})$. First of all, we will prove that it is complete. Let $F \in L^2(\widetilde{Q})$ such that $\langle F, \gamma_n + \kappa_m \rangle = 0$ for every n and m . By the Weil's formula, $m_G = m_K \times m_{G/K}$ provided we renormalize conveniently the Haar measure on G/K . So, using this formula and the fact that $(k, \gamma_n) = 1$ for every $k \in K$ we get for every m and every n

$$\begin{aligned} 0 &= \int_{\widetilde{Q}} F(g) \overline{(g, \gamma_n + \kappa_m)} dm_G(g) \\ &= \int_Q \left(\int_K F(g+k) \overline{(g+k, \gamma_n + \kappa_m)} dm_K(k) \right) dm_{G/K}(\pi(g)) \\ &= \int_Q \left(\int_K F(g+k) \overline{(g, \kappa_m)} \overline{(k, \kappa_m)} dm_K(k) \right) (\pi(g), \gamma_n) dm_{G/K}(\pi(g)). \end{aligned}$$

Fix m . Then, using that $\{\gamma_n\}$ is a Riesz basis for $L^2(Q)$ we get that

$$\overline{(g, \kappa_m)} \int_K F(g+k) \overline{(k, \kappa_m)} dm_K(k) = 0 \quad m_{G/K} - a.e.$$

So, since $\{\kappa_m\}$ is a (countable) orthonormal basis of K , we get that

$$\int_K |F(g+k)|^2 dm_K(k) = \sum_m \left| \int_K F(g+k) \overline{(k, \kappa_m)} dm_K(k) \right|^2 = 0 \quad m_{G/K} - a.e.$$

So, by the Weil's formula we get:

$$\|F\|_{L^2(\widetilde{Q})}^2 = \int_Q \left(\int_K |F(g+k)|^2 dm_K(k) \right) dm_{G/K}(\pi(g)) = 0.$$

Therefore $\{\gamma_n + \kappa_m\}$ is complete.

Now, in order to prove that it is also a Riesz sequence, consider a sequence $\{c_{n,m}\}$ with finitely many non-zero terms. Then

$$\begin{aligned} \left\| \sum_{n,m} c_{n,m} (\gamma_n + \kappa_m) \right\|_{L^2(\tilde{Q})}^2 &= \int_{\tilde{Q}} \left| \sum_{n,m} c_{n,m} (g, \gamma_n + \kappa_m) \right|^2 dm_G(g) \\ &= \int_Q \left(\int_K \left| \sum_{n,m} c_{n,m} (g+k, \gamma_n + \kappa_m) \right|^2 dm_K(k) \right) dm_{G/K}(\pi(g)) \end{aligned}$$

Since $(k, \gamma_n) = 1$ for every $k \in K$, the sum inside the integrals can be rewritten as

$$\sum_{n,m} c_{n,m} (g+k, \gamma_n + \kappa_m) = \sum_m \left((g, \kappa_m) \sum_n c_{n,m} (\pi(g), \gamma_n) \right) (k, \kappa_m).$$

Therefore, using that $\{k_m\}$ is an orthonormal basis of $L^2(K)$ we get

$$\int_K \left| \sum_{n,m} c_{n,m} (g+k, \gamma_n + \kappa_m) \right|^2 dm_K(k) = \sum_m \left| \sum_n c_{n,m} (\pi(g), \gamma_n) \right|^2.$$

So, putting all together

$$\left\| \sum_{n,m} c_{n,m} (\gamma_n + \kappa_m) \right\|_{L^2(\tilde{Q})}^2 = \sum_m \int_Q \left| \sum_n c_{n,m} (\pi(g), \gamma_n) \right|^2 dm_{G/K}(\pi(g)).$$

Finally, since $\{\gamma_n\}$ as a Riesz basis for $L^2(Q)$, there exist $A, B > 0$ such that

$$A \sum_{m,n} |c_{n,m}|^2 \leq \left\| \sum_{n,m} c_{n,m} (\gamma_n + \kappa_m) \right\|_{L^2(\tilde{Q})}^2 \leq B \sum_{m,n} |c_{n,m}|^2,$$

and this concludes the proof. ■

Remark 5.6. Although our version is for Riesz bases, it also holds for orthonormal basis or frames with minor changes. For instance, a proof for orthonormal basis is contained in the proof of Lemma 3 of [6]. ▲

6 Proof of the main result

Finally, using the techniques developed in the previous sections, we provide the proof of our main Theorem 3.5.

(i) **Sampling case:** Since the Haar measure is regular, there exists an open subset V of \widehat{G} such that $\Omega \subseteq V$ and $m_{\widehat{G}}(V \setminus \Omega) \leq \varepsilon$. By Lemma 5.3, there exists a compact subgroup K of \widehat{G} so that \widehat{G}/K is elemental, $m \in \mathbb{N}$ large enough, and $Q_{\lambda_1}, \dots, Q_{\lambda_k} \in \mathcal{D}_K^{(m)}$ are such that

$$\Omega \subseteq \bigcup_{j=1}^k Q_{\lambda_j}^{(m)} \subseteq V.$$

Let Ω_ε be the union of these k quasi-dyadic cubes. Then, by Proposition 5.4, the space $L^2(\Omega_\varepsilon)$ admits a Riesz basis consisting of characters of \widehat{G} (restricted to Ω_ε). Let $\{e_{b_n} \chi_{\Omega_\varepsilon}\}$ denote this basis, and let $J_\varepsilon = \{b_n\} \subseteq G$. Using Theorem 3.4 we get that $\mathcal{D}(J_\varepsilon) = m_{\widehat{G}}(\Omega_\varepsilon) \leq m_{\widehat{G}}(\Omega) + \varepsilon$. Note that $\{e_{b_n} \chi_{\Omega}\}$ is a frame for $L^2(\Omega)$, because it is obtained by projecting a Riesz basis for the bigger space $L^2(\Omega_\varepsilon)$. So, J_ε is a sampling set for PW_Ω .

(ii) **Interpolation case:** Since $m_G(\partial\Omega) = 0$, we can work with the interior of Ω . For the sake of simplicity we will use the same letter for it. Let C be a compact subset of Ω such that $m_{\widehat{G}}(\Omega \setminus C) \leq \varepsilon$. Again by Lemma 5.3, there exists a compact subgroup K of \widehat{G} so that \widehat{G}/K is elemental, $m \in \mathbb{N}$ large enough, and $Q_{\lambda_1}, \dots, Q_{\lambda_k} \in \mathcal{D}_K^{(m)}$ are such that

$$C \subseteq \bigcup_{j=1}^k Q_{\lambda_j}^{(m)} \subseteq \Omega.$$

As before, if Ω_ε is the union of the k quasi-dyadic cubes, then the space $L^2(\Omega_\varepsilon)$ admits a Riesz basis consisting characters of \widehat{G} (restricted to Ω_ε). In this case, the set J^ε consisting of these characters, forms a Riesz sequence in $L^2(\Omega)$. This is equivalent to say that, as points of G , they form an interpolation set for PW_Ω . Since $\mathcal{D}(J^\varepsilon) = m_{\widehat{G}}(\Omega_\varepsilon) \geq m_{\widehat{G}}(\Omega) - \varepsilon$, which concludes the proof.

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